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SET-THEORETICAL PARADOXES, MANY-VALUED LOGIC AND FIXED POINT THEOREMS

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1. INTRODUCTION

The "natural" way of forming a set is to make a collection out of all objects satisfying some common property. In current symbolism this means that we would like to have the validity of the following axiom scheme

$$(1) \qquad \forall x_1 \dots \forall x_n \exists y \forall t (t \in y \iff \varphi(t, x_1, \dots, x_n)) \quad ,$$

where φ is any expression formed from the identity and membership relations by use of the logical connectives.

If this were permissible, the set-theoretic way of life would have been easy, - but life is not simple, i.e. we have the Russel paradox by choosing for φ in (1) the formula $\neg(x \in x)$, which immediately yields the contradiction that $(y \in y) \iff \neg(y \in y)$, where $y = \{x \mid \neg(x \in x)\}$.

The usual (and sensible) way to get out of this dilemma is to restrict the axiom scheme (1) , and then, of necessity, supplement it by various other axioms of set existence, so that one arrives at one of the common versions of axiomatic set theory.

In this paper we shall try another escape route. We will keep the axiom and change the logic, i.e. we do insist that every φ corresponds to a set, but then allow for a change in the usual two-valued classical logic. Of course, we then no longer understand what we talk about, but the formalism is O.K.

2. POSITIVE LOGIC

Negation seems to be at the very root of contradiction. It is essential for the Russel paradox, and according to the usual definition one says that a theory is contradictory if for some φ , both φ and $\neg\varphi$ is provable within the theory. Thus if we throw away the negation sign and keep the rest of logic, everything should be fine. However, if a theory is "strong enough" to contain some fragment of arithmetic, we may express the condition of contradiction without the negation sign, simply by calling a theory contradictory if $0 = 1$ is provable within it.

Let us consider a set-theory based on positive logic and let us assume the axiom scheme (1) and the usual equality axioms. We may then define a set $\emptyset$ by

$$(2) \qquad x \in \emptyset \Leftrightarrow \forall y (x \in y) .$$

Thus $\emptyset$ is the "smallest" set of our universe, and in classical theory it would simply be the empty set. Not so in positive logic! Define R by

$$(3) \qquad x \in R \Leftrightarrow (x \in x \Rightarrow x \in \emptyset) .$$

In classical logic $\neg(x \in \emptyset)$ is provable, hence in this logic $x \in R \Leftrightarrow \neg(x \in x)$, i.e. R is the positive analogue of the Russel set.

We propose to show that $R \in \emptyset$. Substituting R for x in (3) we obtain $R \in R \Rightarrow (R \in R \Rightarrow R \in \emptyset)$, which gives

$$(4) \qquad R \in R \Rightarrow R \in \emptyset .$$

Now, (4) and (3) together give by modus ponens $R \in R$, which again by (4) and modus ponens yields $R \in \emptyset$.

We assume that 0 and 1 are objects of our theory. Hence by (1) we get sets $\{0\}$ and $\{1\}$ by the scheme: $x \in \{y\} \Leftrightarrow x = y$. By (2) and $R \in \emptyset$ we obtain $R \in y$ for all sets y, thus $R \in \{0\}$ and $R \in \{1\}$, i.e. $R = 0$ and $R = 1$. By the equality axioms this implies that $0 = 1$. (And furthermore, $x = y$ for all sets x and y).

By the adopted definition our theory is contradictory, - positive logic is no good.

3. MANY-VALUED LOGIC

The reasoning of section 2 is due to Th. Skolem ((3)) who also investigated the situation within many-valued logic (Skolem ((4))). The purpose of the present paper is to report on some recent investigations into the consistency of the axiom of comprehension (1) within infinite valued logic. In this section we are going to describe the logic and the models so as to make our problem precise.

The logic is a Łukasiewicz many-valued logic having the following primitive symbols. Propositional connectives are $\vee$, $\neg$ and $\Rightarrow$. The quantifier is $\exists$. ($\wedge$, $\Leftrightarrow$ and $\forall$ is definable as usual.) Predicates are $\in$ and $\equiv$ (identity). The class of formulas is inductively defined as usual.

The intended interpretation will be described through the notion of model. A *model* is a pair $M = \langle S, e \rangle$ where S is some set and $e : S^2 \rightarrow [0,1]$. The interval $[0,1]$ will be the range of truth values of the logic $Ł$. An *interpretation* of $Ł$ into M is a map $w : V \rightarrow S$, where V is the set of variables, $v_1, v_2, \ldots, v_n, \ldots$ of $Ł$. The truth-value of each formula φ under the interpretation w will be defined inductively:

$$
(5)\qquad
\begin{array}{rll}
\text{i.} & w(x \in y) = e(w(x), w(y)) , & \\
\text{ii.} & w(x \equiv y) = d_{w(x),w(y)} , & \text{where } d_{aa} = 1 \text{ and } d_{ab} = 0 \\
 & & \text{if } a \neq b , \\
\text{iii.} & w(\neg \varphi) = 1 - w(\varphi) , & \\
\text{iv.} & w(\varphi \vee \psi) = \max(w(\varphi), w(\psi)) , & \\
\text{v.} & w(\varphi \Rightarrow \psi) = \min(1, 1 - w(\varphi) + w(\psi)) , & \\
\text{vi.} & w(\exists x \varphi) = \sup(w'(\varphi)) , &
\end{array}
$$

where $w' : V \to S$ satisfies $w'(y) = w(y)$ for all $y \neq x$. In this way w associated with each formula φ a definite numerical value in the range $0,1$, called the truth-value of φ under the interpretation w of $\mathcal{L}$ in the model $M = \langle S,e \rangle$.

A set Δ of formulas of $\mathcal{L}$ is called s a t i s f i a b l e if there exists a model M and an interpretation w into M such that $w(\varphi) = 1$ for all $\varphi \in \Delta$. Δ is called c o n s i s t e n t if it is satisfiable.

Let Σ be the set of formulas

$$\forall x_1 \ldots \forall x_n \exists y \forall t (t \in y \iff \varphi(t,y,x_1,\ldots,x_n)) ,$$

where φ is an arbitrary formula of $\mathcal{L}$ (note that y is allowed to occur in φ). The problem of the consistency of the axiom of comprehension within $\mathcal{L}$ is simply this: Is the set Σ consistent?

We do not know the answer. The situation seems to be fully summarized in the following theorems.

T h e o r e m A. Let $\varphi(t,y,x_1,\ldots,x_n)$ be defined without quantifiers, then the set Σ_0 of all formulas

$$\forall x_1 \ldots \forall x_n \exists y \forall t (t \in y \iff \varphi(t,y,x_1,\ldots,x_n))$$

is consistent within $\mathcal{L}$.

This was proved by Skolem ((4)) using a combinatorial argument. We will present a new proof below using an extension of the Brouwer fixed point theorem.

T h e o r e m B. Let $\varphi(t,y)$ be an arbitrary formula of $\mathcal{L}$. Then the set Σ_1 of all formulas

$$\exists y \forall t(t \in y \Leftrightarrow \varphi(t,y))$$

is consistent within $\mathcal{L}$ *.*

This was proved by a very simple application of the Brouwer theorem by Chang ((1)) , we will present his proof in the following section.

T h e o r e m C . *Let* $\varphi(t,y,x_1,\ldots,x_n)$ *be a formula of* $\mathcal{L}$ *such that no bound variable* u *of* φ *can occur in the first place of an atomic formula* $u \in v$ *unless* $u = v$ *. Then the set* Σ_2 *of all formulas*

$$\forall x_1 \ldots \forall x_n \exists y \forall t(t \in y \Leftrightarrow \varphi(t,y,x_1,\ldots,x_n))$$

is consistent within $\mathcal{L}$ *.*

This is the "difficult" theorem of Chang ((1)) and the proof is indeed involved. We have recently obtained a similar theorem.

T h e o r e m D . *Let* $\varphi(t,y,x_1,\ldots,x_n)$ *be a formula of* $\mathcal{L}$ *such that* t *can only occur as a variable* u *in the first place of an atomic formula* $u \in v$ *. Then the set* Σ_3 *of all formulas*

$$\forall x_1 \ldots x_n \exists y \forall t(t \in y \Leftrightarrow \varphi(t,y,x_1,\ldots,x_n))$$

is consistent within $\mathcal{L}$ *.*

The restriction on the variable t is easy to state, but quite serious as regard applications of the theorem. The axiom of infinite union is included as the φ in this case is the formula $\exists z(t \in z \wedge z \in x)$. The axiom of powerset is not covered as the φ this time should be the formula $\forall z(z \in t \Rightarrow z \in x)$, and $z \in t$ violates the restriction of the theorem. A non-classical case is included simply by choosing $\neg(t \in y)$ for the formula φ .

4. THE BROUWER FIXED POINT THEOREM

We proceed with an outline of the proof of theorem B. In addition to the Brouwer fixed point theorem the following fundamental result on the logic $\L$ is used. (The proof is far from trivial, using e.g. the ultra-product construction adopted to many-valued logic.)

T h e o r e m . <u>For a set Δ to be consistent in $\L$ it is necessary and sufficient that every finite subset of Δ be consistent.</u>

Therefore let $\varphi_1(t,y), \ldots, \varphi_k(t,y)$ be a finite set of "kernel" formulas for the set Σ_1 . We are going to construct a model in the set $S = \{1,2,\ldots,k\}$. In finite models quantifiers may be replaced by repeated disjunctions or conjunctions, hence introduce a transform $T_k(\varphi)$ by replacing parts $\exists x \, \psi(x)$ by $\psi(z_1) \vee \ldots \vee \psi(z_k)$ and parts $\forall x \, \psi(x)$ by $\psi(z_1) \wedge \ldots \wedge \psi(z_k)$. Each formula $T_k(\varphi_j)$ has at most the variables $t,y,z_1,\ldots,z_k$ free; denote it by $T_k(\varphi_j)(t,y,z_1,\ldots,z_k)$. Define the formulas W_{ij} , $1 \leqslant i,j \leqslant k$ by

$$W_{ij}(z_1,\ldots,z_k) = T_k(\varphi_j)(z_i,z_j,z_1,\ldots,z_k) \quad .$$

Let e be any map of $S^2 \longrightarrow [0,1]$ and define the interpretation w into $M = \langle S,e \rangle$ by $w(z_i) = i$ (and arbitrary on $x \neq z_i$, $i = 1,\ldots,k$) . Denote by $W^{\star}_{ij}$ the truth-value of W_{ij} under the interpretation w . We then have a map $f : [0,1]^{k^2} \longrightarrow [0,1]^{k^2}$ defined by

$$pr_{ij} \circ f(e) = W^{\star}_{ij}$$

Let $[0,1]^{k^2}$ have the usual topology, it is easily seen from (5)i.–v. that f is continuous. Hence by the Brouwer fixed point theorem there is a point e_o such that $f(e_o) = e_o$. Then in the model M let $e = e_o$, which implies that $w(z_i \in z_j) = e_o(i,j) = W^*_{ij} = w(T_k(\varphi_j)(z_i,z_j,z_1,\ldots,z_k))=$ $w(\varphi_j(i,j))$. Hence

$$w(z_i \in z_j \Leftrightarrow \varphi_j(z_i,z_j)) = 1 .$$

For a given φ_j this equality holds for z_j and all z_i, which by the rules of $\mathcal{L}$ gives

$$w(\exists y \forall t(t \in y \Leftrightarrow \varphi_j(t,y)) = 1 ,$$

and this by the definition of satisfiability proves theorem B.

5. APPLICATION OF AN EXTENDED FIXED POINT THEOREM

In this section we present a proof of theorem A. The proof of theorem D will not be given, it will appear in a forthcoming paper in Mathematica Scandinavica.

As preparation for the proof proper we present a simple extension of the Brouwer fixed point theorem. This extension is almost identical to a lemma given in Dunford and Schwartz: Linear Operators, vol. I, p. 453, but for convenience we repeat the short argument.

Let E be a countable product of intervals $I = [0,1]$ given the product topology. Then E is a compact metric space in the metric defined by

$$d(x,y) = \sum_{m=1}^{\infty} \frac{|x_m - y_m|}{2^m} .$$

It will be shown that each continuous map $f : E \longrightarrow E$ has the fixed point property.

To do this define the "projections" $\pi_n(x) = x'$ where $x'_i = x_i$ if $i \leqslant n$ and $x'_i = 0$ if $i > n$. The subset $E_n = \pi_n(E)$ of E has (in the induced topology) the fixed point property by Brouwer's theorem. Let $f_n = \pi_n \circ f \circ \mathrm{inj}$, this map is obviously continuous of E_n into E_n, hence has a fixed point $y_n \in E_n \subsetneq E$. As E is compact the sequence y_n contains a convergent subsequence y_{n_i}. Let $y_o = \lim y_{n_i}$, we propose to show that $f(y_o) = y_o$. To this end consider the inequality

$$d(f(y_o),y_o) \leqslant d(f(y_o),f(y_{n_i})) + d(f(y_{n_i}),f_{n_i}(y_{n_i})) + d(y_{n_i},y_o) .$$

Here the first and last term of the right hand sum can be made arbitrary small as f is continuous and $y_{n_i} \longrightarrow y_o$. Further

$$pr_j \circ f(y_{n_i}) = pr_j \circ f_{n_i}(y_{n_i}) , \quad j \leqslant n_i ,$$

thus

$$d(f(y_{n_i}),f_{n_i}(y_{n_i})) = \sum_{m=n_i+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^{n_i}} ,$$

hence the middle term of the sum can also be made arbitrary small by choosing i large enough. Therefore, $f(y_o) = y_o$.

The formulas of theorem A can be enumerated in a sequence $\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots$. With each φ_m we may associate a number n_m such that φ_m can be written $\varphi_m(t,y,x_1,\ldots,x_{n_m})$. It is no restriction to assume that $n_m \geqslant 1$. Further for any $n \geqslant 1$, let λ_n denote a bijection of N^n onto N , where N is the set of natural numbers.

With every $e \in E$ we may associate a model $M = \langle N,e \rangle$ of $\mathcal{L}$ by defining $e(i,j) = pr_{\lambda_2(i,j)}(e)$. By use of this model a map $f : E \rightarrow E$ will be introduced by the following coordinate equations:

$$pr_{\lambda_2(i,j)} \circ f(e) = w(\varphi_m(t,y,x_1,\ldots,x_{n_m})) ,$$

where w is any interpretation of $\mathcal{L}$ into M such that $w(t) = i$, $w(y) = j$, and $w(x_1) = k_1,\ldots,w(x_{n_m}) = k_{n_m}$, where $m,k_1,\ldots,k_{n_m}$ are the unique numbers such that

$$j = \lambda_2(m,\lambda_{n_m}(k_1,\ldots,k_{n_m})) .$$

To show that the map f defined above is continuous it is sufficient to prove that each coordinate map $pr_n \circ f : E \rightarrow I$ is continuous. But the value of $pr_n \circ f(e)$ is equal to $w(\varphi_m(t,y,x_1,\ldots,x_{n_m}))$ for some m and w , and this truth-value is determined by a finite number of coordinates of e , a fact which taken in conjunction with the definition of an interpretation, immediately yields the continuity of the map $pr_n \circ f(e)$. This argument also shows why we cannot allow bound quantifiers in a kernel formula φ_m , because then $w(\varphi_m)$ would in general depend on an infinite number of coordinates of e , hence the map need not be continuous.

By the above fixed point lemma the map f has a fixed point, say e_o . Define a model $M_o = \langle N,e_o \rangle$, where N is the set of natural numbers and $e_o(i,j)$ is given by

$$e_o(i,j) = pr_{\lambda_2(i,j)}(e_o) .$$

Let w be any interpretation of $\mathfrak{L}$ into M_o, it will be shown that

$$w(\forall x_1 \ldots \forall x_{n_m} \exists y \forall t(t \in y \iff \varphi_m(t,y,x_1,\ldots,x_{n_m}))) = 1 ,$$

for all m. This will complete the proof of theorem A. But the truth of this equality is almost immediate by the definition of an interpretation. Assume that $w(x_1) = k_1,\ldots,w(x_{n_m}) = k_{n_m}$. Define w' equal to w for all variables different from y and set $w'(y) = \lambda_2(m, \lambda_{n_m}(k_1,\ldots,k_{n_m}))$. Then let w'' be any interpretation agreeing with w' except possibly for the variable t. Let $w''(t) = i$, for some $i \in N$. We must show that

$$w''(t \in y \iff \varphi_m(t,y,x_1,\ldots,x_{n_m})) = 1 .$$

But this is the case if and only if $w''(t \in y)$ and $w''(\varphi_m(t,y,x_1,\ldots,x_{n_m}))$ are equal. But

$$w''(t \in y) = e_o(i,j) = pr_{\lambda_2(i,j)}(e_o) ,$$

and

$$w''(\varphi_m(t,y,x_1,\ldots,x_{n_m})) = pr_{\lambda_2(i,j)}(f(e_o)) .$$

This concludes the proof, because e_o is a fixed point of the map f.

References

((1)) C.C. Chang: The axiom of comprehension in infinite valued logic. To appear in Math. Scand.

((2)) Dunford, Schwartz: Linear operators, Part I. New York 1958.

((3)) Th. Skolem: A remark on a set theory based on positive logic. Kgl. N. Vid. Selsk. Forh. vol. 25, 1952.

((4)) Th. Skolem: Bemerkungen zum Komprehensionsaxiom. Z. Math. Logik und Grund. Math., vol. 3, 1957.